Topological Defects in a Spin 1 BEC

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Introduction

Topological defects and states have been observed in a wide array of systems including superfluid helium films, solid state systems and ultracold atomic experiments. This paper will review the theory of topological defects in ordered media, specifically in ultracold atomic spin 1 systems. We will find a nematic phase in an anti-ferromagnetic spin 1 BEC, similar to the one in liquid crystals. Lastly, we will discuss experimental results in Rb/Na BEC experiments, where the ground state has total spin 1 and topological defects have been directly observed.

I. MOTIVATION: TOPOLOGY IN SPIN SYSTEMS

The application of topology to physics has been an intense focus of research in the last few decades. Topological states are robust against imperfections and noise and have potential applications for fault tolerant quantum computing and information storage.

We will specifically consider topological defects in the ordered phase of spin systems. In ordered states, where an order parameter characterizes a phase of matter, there can sometimes be excitations that cannot continuously dissipate, and are thus topologically protected. These defects are general features of ordered systems.

To consider another example, take the XY model. This is the theory of a complex scalar field with a U(1) symmetry, which we can visualize as arrows or spins living in a 2D plane. There are spin configurations, however, that cannot continuously evolve to the uniform ferromagnetic ground state. If spins are initially placed pointing clockwise along a circle, for example, then the spins cannot align themselves continuously. [1] So if one excites such a spin configuration (a defect in the ferromagnet), it will be topologically protected. In 2D, although the Mermin-Wagner theorem forbids long range order from a spontaneously broken continuous symmetry at finite temperature, there can be quasi-long range order in the XY model. Above a critical temperature, however, free defects can propagate and scramble the spin order. This is called a Kosterlitz-Thouless (KT) transition. [2]

To understand the possible topological defects of a system, we will need to introduce two mathematical concepts: the order parameter space of the system, and the homotopy groups of that space.

A. Order Parameter Manifold

The order parameter space is defined as M = G/H, where G is the total symmetry of the system, and H is the unbroken subgroup of symmetries in the ground state. [1] For the XY model, G = U(1), H = $\mathbb{I}$, so the order parameter space is U(1).

B. Homotopy Groups

To determine which defects are allowed in a given parameter space, M, we must consider n-dimensional closed surfaces embedded in M that cannot continuously deform into each other. [1] These will tell us the topologically protected spin defects and determine the n-th homotopy group, π_n(M). The homotopy group for n = 1 is called the fundamental group. [3]

Returning to the XY model, we may consider $\pi_1(U(1) \cong S^1)$, the fundamental group of a circle. We can wind a loop around a circle an integer number of times, and all these loops cannot be deformed into each other. Therefore, the

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fundamental group is $\mathbb{Z}$ for the number of times a loop is wound around the circle. Spin arrangements with non-zero winding number in the XY model are therefore topologically protected.

II. SPIN 1 ATOMS

Many ultracold atomic experiments use atoms with total spin 1. For example, in ^{87}Rb and ^{23}Na, the electron spin is $\frac{1}{2}$ and the nuclear spin is $\frac{3}{2}$ giving total F = 1 in the ground state. For these atoms, we must consider the spin degrees of freedom along with the familiar phase.

The spin 1 representation of SU(2) can be constructed in the following way from the fundamental spin 1/2 representation ($\square$):

$$\frac{1}{2} \otimes \frac{1}{2} = 3 \oplus 1 \text{ or } \square \otimes \square = \square\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}$$

Note that the spin 1 representation is the first term in the direct sum while the second term is the trivial representation or singlet. We can immediately see that the Spin 1 representation is 3 dimensional and symmetric. Therefore, we can write the effective interaction between these atoms as

$$c\psi_a^\dagger \psi_{a'}^\dagger \vec{F}_{ab} \cdot \vec{F}_{a'b'} \psi_{b'} \psi_b$$

where $\vec{F}$ is a basis of generators for the irreducible spin 1 representation of su(2), ψ_a is the annihilation operator for an atom in spin state a (0,1, or -1), and the sign of the constant c determines whether the ground state is ferromagnetic or anti-ferromagnetic. [4] Experimentally, sodium has an anti-ferromagnetic ground state, while rubidium has a ferromagnetic ground state. [5]

A. Order Parameter Space for a Spin 1 Gas

A spin 1 gas has a spin symmetry SO(3) from the 3 m levels, as well as the familiar global U(1) phase for a BEC. Together this gives a total symmetry group G=SO(3)×U(1). The unbroken subgroup depends on whether the system is ferromagnetic or anti-ferromagnetic (AFM).

1. *Ferromagnetic Case*

In the ferromagnetic case, there is a nonzero magnetization. One way to achieve this is to have all atoms in m = 1. There is a U(1) unbroken symmetry in the ferromagnetic ground state given by rotations around the quantizing axis by α offset with a change in superfluid phase by $-\alpha$. Therefore the parameter space is: [4]

$$(SO(3) \times U(1))/U(1) \cong SO(3).$$

This state has no order at finite temperature, but the ferromagnetic state, as we will see later, can undergo a KT transition in a magnetic field. [6]

2. *Anti-ferromagnetic Case*

The more interesting case is the anti-ferromagnetic or polar gas, which will have total magnetization as 0. One obvious way to have total magnetization 0 is to have all the atoms in the m=0 state. Without an external field however, we can rotate the quantization axis at will, and still have total magnetization = 0. Thus, we can write a general state in the m = -1,0,1 basis in terms of Euler angles (α,β,γ) and superfluid phase, θ as [4]:

$$e^{i\theta} \begin{pmatrix} -\frac{1}{\sqrt{2}} e^{-i\alpha} \sin(\beta) \\ \cos(\beta) \\ \frac{1}{\sqrt{2}} e^{i\alpha} \sin(\beta) \end{pmatrix}$$

Note there is a SO(2) symmetry in this state since γ can have an arbitrary value (i.e. we can rotate around the quantization axis). There is an additional Z_2 symmetry by sending α to $\alpha + \pi$, β to $\pi - \beta$ and θ to $\theta + \pi$ that we can picture as a reflection in spin space. Together, rotations and reflections give an O(2) symmetry. Therefore, the order parameter space is M = (SO(3)×U(1))/O(2). We can gain insight by noting that [4]:

$$(SO(3) \times U(1))/O(2) \cong (S^2 \times U(1))/Z_2.$$

Therefore, this order parameter space can be described by identifying a point on the 2-sphere with a superfluid phase θ, with the opposite point on the sphere and phase $\theta + \pi$.

In liquid crystals, there is a similar phase called a nematic phase when elongated molecules align since one side of the molecule is identical to the other, giving a discrete reflection symmetry Z_2. However, for liquid crystals, there is no global phase. The initial symmetry group of the molecule is just SO(3), the unbroken symmetry when molecules align is again H = $O(2)$, (for rotations and reflections), and so the order parameter space is:

$$SO(3)/O(2) \cong \mathbb{R}P^2 \cong S^2/Z_2$$

We can therefore call our anti-ferromagnetic phase nematic. [7]

B. Fundamental Group of a Nematic Spin 1 Gas

To determine the fundamental group of M, we must consider inequivalent closed paths in M. Such closed loops can either do a half loop around U(1) and move to the opposite side of sphere, or do a full loop around U(1) and stay at the same point as the sphere, which is the same as the doing the half loop twice. This implies

$$\pi_1((SO(3) \times U(1))/O(2)) = \mathbb{Z}$$

for the number of half loops we do. [4] A spin defect that does a half-loop in phase is called a half-vortex and is topologically stable. The second homotopy group is also $\mathbb{Z}$. [4]

C. Spin 1 AFM gas in a Magnetic Field

In real experimental systems, we must consider the effects of magnetic fields. Although the linear Zeeman shift won't change the relative energy of |m|=1 and m=0, a quadratic Zeeman shift will energetically prefer one state over the other. [4] [5] We must consider both cases. Our total symmetry will now be $SO(2) \times U(1) \cong U(1) \times U(1)$ since the the magnetic field fixes the quantization axis. We will only consider an anti-ferromagnetic gas, but a similar analysis holds for the ferromagnetic case.

If |m|=1 is favored, the ground state (written explicitly on the previous page) has $\beta = \pi/2$. This state only has a discrete Z_2 symmetry, α can only be sent to itself and $\alpha + \pi$ therefore there is a

$$(U(1) \times U(1))/Z_2$$

order parameter manifold. For this case, it is easy to see there are still half-vortices. [4]

If m=0 is favored, the ground state has β=0, and α can take an arbitrary value giving a U(1) unbroken symmetry. Therefore M = U(1) for this case just like in the XY model and there will be a normal KT transition with regular vortices. [4]

III. EXPERIMENTAL RESULTS

Ultracold atoms have tunable interactions, and offer a virtually impurity free system for the study of many-body physics. Spinor condensates using ^{87}Rb and ^{23}Na have been used to observe a number of topological effects. We will discuss just a few experimental results, but there are many others and this remains an active area of research.

A. AFM BEC Experiments

Experiments with ultracold sodium atoms have confirmed it has an antiferromagnetic ground state, measured the effect of magnetic fields, and observed topological defects in 2D. We discuss two such experimental results below.

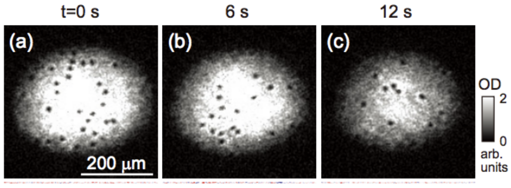

FIG. 1: Figure from [9] showing vortices appear after stirring a Na BEC with a laser beam. Note that the vortices are stable for seconds, due to their topological protection.

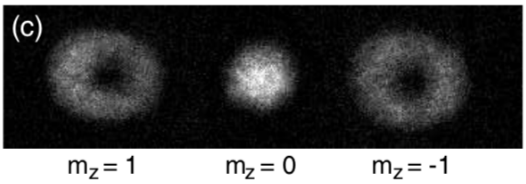

FIG. 2: Figure from [10] showing atoms after a short time of flight (so that the atoms approximately keep their in situ positions) and after a Stern Gerlach kick to separate the spin states (m = -1,0,1 shown left to right). Note that the m = 1,-1 atoms wrap around the m=0 atoms.

1. Quench Between Ground States

The Shin group used a sodium BEC in a magnetic field to explore the two possible ground states by quenching from the m=0 preferred state to the m=1/-1 preferred anti-ferromagnetic states using a microwave pulse. The group observed turbulence as the atoms flip to m=1,-1, and the emergence of localized defects in the gas, which they interpreted as half-vortices. [8] In another paper, the same group observed regular vortices after stirring a Na BEC with a laser beam (shown in Figure 2). [9]

2. Other Topological Defects

Topological defects other than vortices have also been observed in a polar BEC. [10] Beginning with the atoms in a optical dipole trap, the Shin group also turned on a quadrapole magnetic field and a bias field, following a protocol used earlier other experiments. [11] By quickly sweeping the bias field, they are able to create a topological spin defect since atoms near the edge of the quadrapole magnetic field are able to precess their spins around the changing field, but in the center atoms cannot since the field changes too quickly. Thus, there is a spatially dependent spin pattern. By controlling the speed of the magnetic field ramp, the group was able to create a topological spin defect called a 2D skyrmion (corresponding to non-trivial element of π_2) shown in Figure 2. [10] The spin defect has a lifetime of 100ms demonstrating its topological protection.

B. KT Transition

Returning to the ferromagnetic case, Jean Dalibard's group utilized a quasi-2D Rb BEC in a magnetic field to observe a KT type transition. [6] By superimposing two retro-reflected beams on a 3D condensate, the group separated the atoms into two quasi-2d condensates a few microns apart. When the trap was turned off, the condensates expanded and interfered, and that interference could be measured through absorption imaging. At low temperature, the phase of the interfering condensates was uniform, but at high temperature, the phase order oscillated, and in some places jumped suddenly, indicative of being above a KT transition, where free vortices can scramble phase order (see Figure 3). [6]

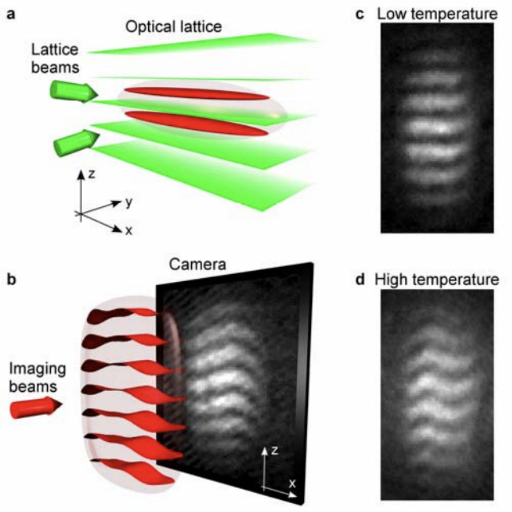

FIG. 3: Figure from [6] showing the experimental method (a,b), and different interference patterns at high and low temperature (c,d) from quasi-2D Rb BEC's. At high temperature, there is clearly less phase order than at low temperature, indicative of a KT transition.

Conclusion

Ordered systems can exhibit very different topological defects, depending on their order parameter space. In an anti-ferromagnetic spin 1 BEC, as discussed above, there is a nematic phase which can contain half-vortices. Similar to the XY model, the transition from this nematic phase to a trivial state will be at a finite transition temperature when half-vortices unbind. Some features of spin 1 condensates like the jump in superfluid density at the critical temperature remain to be observed. There are therefore exciting opportunities for groups to continue to study spinor condensates, including with atoms that have total F > 1, not discussed in this paper.

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