Clock Model

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Classical 2D XY model and clock model are discussed. The phase diagram of the clock model reveals rich phenomena depending on $p$, the ways to discretize the angle. When $2 \leq p \leq 4$, there are two Ising type phases. When $p \geq 5$, an intermediate phase emerge, which is related to the Berezinskii–Kosterlitz–Thouless (BKT)-type transitions. The criteria of the onset of intermediate phase is discussed based on duality. An experimental setup is mentioned.

REVIEW OF 2D XY MODEL

The 2D XY model offers us a nice platform to study Berezinskii–Kosterlitz–Thouless (BKT)-type transitions [1]. The 2D XY model consists of a two-dimensional lattice of classical spins rotating in a plane with nearest-neighbour interaction. In the high temperature regime the system is in a disordered, paramagnetic phase with an exponentially decaying correlation function. When the temperature goes below some critical temperature $T_k$, the correlation function of the system decays following a power law with the critical exponent behaves non-universally, depending on the temperature. In this way every temperature below $T_k$ behaves as if a critical point. Kosterlitz proposed that such phase transition can be understood from topological excitations, or namely, vortices. When the temperature is below $T_k$, the vortices are bound together. When temperature reaches $T_k$, vortices start to unbind. Both of these two phases are not ordered since it can be proved that the breaking of a continuous symmetry is forbidden in 2 dimensions.

The partition function of the 2D XY model is

$$Z = \int_{-\pi}^{\pi} \prod_i d\theta(i) \exp \left( \beta \sum_{i,\mu} \cos[\theta(i) - \theta(i + e_\mu)] \right)$$

This is similar to the case of 2D XY model, apart from the fact that now the angle $\theta(i)$ are discrete and countable. When $p$ goes to infinity, we are supposed to recover the result of 2D XY model. When $p$ equals to 2, this is exactly the Ising model, where there is a critical temperature $T_2$. Below $T_2$ we have an ordered, ferromagnetic phase while above $T_2$ we have a disordered paramagnetic phase. The clock model has a $Z_p$ symmetry, which is a subgroup of $U(1)$.

The clock model has pretty interesting phase diagrams. For $2 \leq p \leq 4$, there are two phases, one is the high temperature disordered phase and the other is the low temperature ordered state, separated by a second order phase transition. It can be proved that $p = 4$ case is identical to two uncoupled Ising models. As for $p \geq 5$, there is an intermediate phase which separates the ferromagnetic and

FIG. 1. Two dimensional classical XY model. Figure taken from [2].

CLOCK MODEL

The clock model is an approximation to the 2D XY model and provides a testbed for studying various critical phenomena. In 2 dimensions, the clock model is described by a set of $p$ discretized angle $\theta(i)$:

$$\theta(i) = \frac{2\pi n(i)}{p}, n(i) = 0, 1, \cdots, p - 1$$

The partition function in this case is given by

$$Z = \sum_{n(i)=0}^{p-1} \exp \left( \beta \sum_{i,\mu} \frac{2\pi}{p} [n(i) - n(i + e_\mu)] \right)$$

In the above expression, $i$ denotes lattice site and $\mu = 1, 2$ represent lattice unit vectors. The 2D XY model displays a global continuous $U(1)$ symmetry, which means the model is invariant when all the spins in the system is simultaneously rotated by the same angle. The figure of 2D XY model is shown below.
paramagnetic phase. This intermediate phase is characterized by a power-law behaviour of correlation function, which represents the BKT type transition we mentioned. When \( p \) goes to infinity the broken-symmetry phase will disappear and we will recover the 2D XY model. The phase diagram plot is below.

The function \( \exp(f_\beta) \) has the Fourier representation

\[
\exp[f_\beta] = \sum_{l = -\infty}^{\infty} \exp(-\frac{l^2}{2\beta}) \exp(ilx)
\] (6)

Substituting Eq.6 to Eq.5 and summing over \( n(i) \), we get

\[
Z^V(\beta) = \sum_{l(i)=0}^{p-1} \prod_{i} \delta_{\nu,\mu} \exp[-\frac{1}{2\beta} \sum_{i,\mu} l^2_{\mu}(i)]
\] (7)

The constraint \( \nabla_\mu l_\mu = 0 \pmod{p} \) can be solved by

\[
l_\mu(i) = \epsilon_{\nu\mu} [\nabla_\nu l(i) + pm_\nu(i)]
\] (8)

where \( m_\nu(i) \) is also an integer field.

Plug into Eq.7 we get

\[
Z^V(\beta) = \prod_{l(i)=0}^{p-1} \exp[-\frac{1}{2\beta} (\nabla_\mu l + pm^2)]
\] (10)

From Eq.4 we have

\[
Z^V(\beta) = \prod_{l(i)=0}^{p-1} \exp[f_{\beta'}(\frac{2\pi}{p} \nabla_\mu l)] = Z^V(\beta')
\] (11)

where

\[
\beta' = \frac{p^2}{4\pi^2\beta}, \quad T' = \frac{4\pi^2}{tp^2}
\] (12)

We can see that this model is self-dual. Assuming there’s only two Ising-type phases for any finite \( p \), this implies that the transition occurs at the self-dual point \( T_p = \frac{2\pi}{p} \).

As \( p \) increases, \( T_p \) will approach 0 rather than \( T_k \). If you stick with the two phase picture, then for any finite \( p \) and \( T > \frac{2\pi}{p} \), the system is in disordered phase. When \( p \) goes to infinity, somehow for \( T < T_k \) it becomes massless (i.e., inverse of the correlation length) while for \( T > T_k \) it has a finite mass. However this picture is wrong. When \( p \) is finite and \( T < T_k \), then the system is in disordered state with an exponentially decaying correlation function.

There’s an inequality for the Villain model where

\[
\langle \cos[\theta(0) - \theta(x)] \rangle_p \geq \langle \cos[\theta(0) - \theta(x)] \rangle_{XY}
\] (13)

which implies that this case is not right. So we need to discard the two phase picture. Once the self dual point \( T_p = \frac{2\pi}{p} \) gets below \( T_k \) then the third phase must occur. So the condition to have 3 phases is

\[
p > \frac{2\pi}{T_k}
\] (14)
From the reference [6], $T_k = 1.35$, so we have three phases at least for $p > 4$.

Clock model has long been seen as a theoretical model. But actually there are experimental realizations in Coulomb crystals, which are formed of interacting ions. Below is a setup how they realize a six-state clock model.

To conclude, in this report we review the 2D XY model with continuous symmetry and explore the phase diagram of the clock model with discrete symmetry. Depending on how we discretize the angle, the clock model can have different phase diagrams, which opens up a new playground for studying various critical phenomena.

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